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STEADY-STATE RESPONSE OF A FINITE BEAM ON A PASTERNAK-TYPE FOUNDATION

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Abstract—A uniform Bernoulli–Euler beam of finite length is supported by a Pasternak-type foundation and subjected to a harmonic force $F = F_0 e^{j\omega t}$, concentrated at the midpoint. The influence of the "shear layer" is exhibited by comparison with the response of a beam supported by a Winkler-type foundation. At the first resonant frequency, unbounded values occur for the bending moment, which are not expected according to Winkler's hypothesis. Observations are made on the influence of damping and inertia of the foundation. Analytical expressions as well as frequency-response curves are presented for the beam deflections and the bending moments.

NOTATION

A	cross sectional area of the beam
B_1, B_2, B_3, B_4	notations according to equation (27), (27'), (28) and (28')
$C_1, C_2, C_3, C_4, C_5, C_6$	integration constants
E	dynamic modulus of elasticity of beam material
F	exciting concentrated force
G_0	shear foundation modulus
I	moment of inertia of the cross-section of the beam
M	bending moment
Q	total shear force (beam + foundation)
R_f	concentrated foundation pressure
T	shear force in beam
$\alpha, \beta, \varepsilon, \psi, \lambda, \mu$	notations (see Table 1)
γ	foundation characteristic, notation according to equation (15)
ξ	dimensionless coordinate
δ_k, δ_G, η	damping factors
v	abbreviation according to equation (8)
ϱ	mass density of the beam material
ω	forcing frequency
Ω	dimensionless frequency
$\phi_1, \phi_2, \phi_3, \phi_4$	Cauchy-type functions
ζ	notation according to equation (51)
j	$= \sqrt{(-1)}$
k	Winkler's foundation modulus
l	half-length of the beam
m, n	notations according to equations (6) and (7)
q	restoring force from the foundation
v, y	beam deflections
v_f, y_f	deflections of the free surface of the foundation
t	time
x	position along beam and/or foundation

1. INTRODUCTION

THE dynamic response of beams continuously supported by deformable media is of interest in many fields of engineering. The steady-state response problem is most frequently encountered in practical problems concerning vibration isolation, using carpets and layers of high polymers or rubber-like materials.

Usually, the subgrade is replaced either by a Winkler-type elastic foundation [1], or by an isotropic semi-infinite elastic continuum [2]. However, it was shown by Kerr [3] and Soldini [4] that the behaviour of a large class of foundation materials occurring in practice cannot be described by such models. Thus, one often resorts to the so called "approximation of second order" of the restoring force from the foundation [5], namely, to "generalized" foundations, characterized by two moduli [6–8]. The basic assumption is that the foundation reaction at any point has two components; one is proportional to the beam deflection at that point, the other to the curvature of the deflection surface of the subgrade (the latter component entering with a negative sign). The mechanical implication of this hypothesis is that the supporting medium can be replaced by a set of vertical parallel linear springs, which are not independent as for the Winkler model, but have the ends connected by a "shear layer" (Fig. 1), i.e. a beam consisting of incompressible vertical elements, which deform only by transverse shear (Pasternak foundation [7]). Similar behaviour is exhibited by the Wieghardt-type foundation [9] as was recently shown by Ylinen [10] and Capurso [11].

Free vibrations of a Timoshenko beam on a Pasternak foundation have been studied by Menditto [12]. In the present paper, a comparative analysis of the steady-state response of a beam resting on a Pasternak-type and on a Winkler-type foundation is presented. It is shown that the influence of the "shear layer" is most pronounced over the range of low forcing frequencies, where—for the bending moments— the character of the responses is completely different, so that Winkler's hypothesis—the common approach of the problem—leads to great errors just in the frequency range of practical interest.

Freudenthal and Lorsch [13]—for static problems, and Kenney [14], Mathews [15], Achenbach and Sun [16]—for dynamic problems, have introduced linear viscoelastic foundation models, replacing the springs from Winkler's foundation by viscoelastic elements. Kerr [3] proposed some Pasternak-type viscoelastic foundations.

2. EQUATION OF MOTION

The differential equation of the transverse vibration of a flexibly supported Bernoulli–Euler beam, with constant cross-sectional area A, moment of inertia I, Young's modulus E and mass density ϱ is

$$EI\frac{\partial^4 y}{\partial x^4}+\varrho A\frac{\partial^2 y}{\partial t^2}=f(x,t)-q(x,t) \tag{1}$$

where y is the beam deflection at section x and time t, $f(x,t)$ is the impressed force distribution, and $q(x,t)$ is the restoring force from the foundation.

The parameters of the Pasternak foundation are the spring constant k and the shear foundation modulus G_0, so that

$$q(x,t)=ky-G_0\frac{\partial^2 y}{\partial x^2}. \tag{2}$$

Permanent and smooth contact between beam and foundation is assumed.

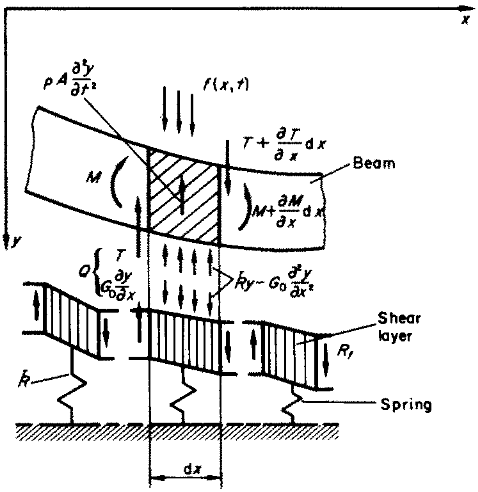

FIG. 1.

Considering $f(x, t) = 0$ (for free of load intervals), equation (1) and (2) yield

$$EI\frac{\partial^4 y}{\partial x^4}-G_0\frac{\partial^2 y}{\partial x^2}+ky+\varrho A\frac{\partial^2 y}{\partial t^2} = 0. \tag{1a}$$

Assuming a steady-state solution

$$y(x, t) = v(x)\,\mathrm{e}^{j\omega t}, \tag{3}$$

where ω is the forced frequency, and introducing the dimensionless coordinate

$$\xi = \frac{x}{l}, \tag{4}$$

where l is the half-length of the beam, equation (1a) can be written as

$$\frac{\mathrm{d}^4 v}{\mathrm{d}\xi^4}-4m^2\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2}+4n^4 v = 0 \tag{5}$$

where

$$m = \left(\frac{G_0 l^2}{4EI}\right)^{\frac{1}{2}} \tag{6}$$

$$n = \nu(1-\Omega^2)^{\frac{1}{4}}, \tag{7}$$

in which

$$\nu = \left(\frac{kl^4}{4EI}\right)^{\frac{1}{4}}, \tag{8}$$

and Ω is the dimensionless forced frequency defined as

$$\Omega = \frac{\omega}{\omega_0}, \tag{9}$$

where ω_0 is the bouncing frequency of the rigid beam on a Winkler foundation

$$\omega_0 = \left(\frac{k}{\varrho A}\right)^{\frac{1}{2}}. \tag{10}$$

The general solution of the homogeneous equation (5) can be written in the form

$$v(\xi) = C_1\phi_1(\xi)+C_2\phi_2(\xi)+C_3\phi_3(\xi)+C_4\phi_4(\xi), \tag{11}$$

where $C_1, \ldots, C_4$ are integration constants (initial parameters) and $\phi_i(\xi)$ are Cauchy-type functions, whose expression depends on the relative values of the quantities m and n (see Table 1). For $\xi = 0$, together with their first three derivatives, these functions form a unit matrix

$$\left.\begin{array}{llll} \phi_1(0) = 1 & \phi_2(0) = 0 & \phi_3(0) = 0 & \phi_4(0) = 0 \\ \phi_1'(0) = 0 & \phi_2'(0) = 1 & \phi_3'(0) = 0 & \phi_4'(0) = 0 \\ \phi_1''(0) = 0 & \phi_2''(0) = 0 & \phi_3''(0) = 1 & \phi_4''(0) = 0 \\ \phi_1'''(0) = 0 & \phi_2'''(0) = 0 & \phi_3'''(0) = 0 & \phi_4'''(0) = 1 \end{array}\right\}. \tag{12}$$

In Table 1, the first five cases refer to the problem of a beam on a Pasternak foundation and the last three ones to that of the beam on a Winkler foundation. In the last column of the table substitutions are given with whose help the corresponding functions $\phi_i(\xi)$ can be deduced from those of a previous case.

For the unloaded surface of the foundation, outside a finite beam, we have $q(x, t) = 0$, and from (2) the equation of motion of the free surface then reduces to

$$G_0\frac{\partial^2 y_f}{\partial x^2}-ky_f = 0, \tag{13}$$

where y_f is the foundation deflection.

Making use of notations similar to (3) and (4), (13) becomes

$$\frac{\mathrm{d}^2 v_f}{\mathrm{d}\xi^2}-\gamma^2 v_f = 0 \tag{14}$$

in which

$$\gamma = \left(\frac{kl^2}{G_0}\right)^{\frac{1}{2}}. \tag{15}$$

The general solution of equation (14) is

$$v_f(\xi) = C_5\,\mathrm{e}^{\gamma\xi}+C_6\,\mathrm{e}^{-\gamma\xi}, \tag{16}$$

where C_5 and C_6 are integration constants.

3. FREE-FREE BEAM WITH A CONCENTRATED FORCE AT THE MIDPOINT

Let us consider a uniform finite beam, supported by a Pasternak foundation and with a sinusoidally varying force $F = F_0\,\mathrm{e}^{j\omega t}$ concentrated at the middle point (Fig. 2). The origin of the coordinates is taken at the driving point. Because of symmetry, only the right side of the beam is considered. The boundary conditions are that at the origin the slope

TABLE 1.

Case	Notation	$\phi_1(\xi)$	$\phi_2(\xi)$	$\phi_3(\xi)$	$\phi_4(\xi)$	Substitution
I	$m > 0$, $n^4 > 0$, $m > n$; $\lambda = (m^2+n^2)^{\frac{1}{2}}+(m^2-n^2)^{\frac{1}{2}}$, $\mu = (m^2+n^2)^{\frac{1}{2}}-(m^2-n^2)^{\frac{1}{2}}$	$\frac{1}{\mu^2-\lambda^2}(\mu^2 \cosh \lambda\xi - \lambda^2 \cosh \mu\xi)$	$\frac{1}{\mu^2-\lambda^2}\left(\frac{\mu^2}{\lambda} \sinh \lambda\xi - \frac{\lambda^2}{\mu} \sinh \mu\xi\right)$	$\frac{1}{\mu^2-\lambda^2}(\cosh \mu\xi - \cosh \lambda\xi)$	$\frac{1}{\mu^2-\lambda^2}\left(\frac{1}{\mu} \sinh \mu\xi - \frac{1}{\lambda} \sinh \lambda\xi\right)$	
II	$m > 0$, $n^4 > 0$, $m < n$; $\alpha = (m^2+n^2)^{\frac{1}{2}}$, $\beta = (n^2-m^2)^{\frac{1}{2}}$	$\cosh \alpha\xi \cos \beta\xi - \frac{\alpha^2-\beta^2}{2\alpha\beta} \sinh \alpha\xi \sin \beta\xi$	$\frac{3\beta^2-\alpha^2}{2\beta(\alpha^2+\beta^2)} \cosh \alpha\xi \sin \beta\xi + \frac{3\alpha^2-\beta^2}{2\alpha(\alpha^2+\beta^2)} \sinh \alpha\xi \cos \beta\xi$	$\frac{1}{2\alpha\beta} \sinh \alpha\xi \sin \beta\xi$	$\frac{1}{2\alpha\beta(\alpha^2+\beta^2)}(\alpha \cosh \alpha\xi \sin \beta\xi - \beta \sinh \alpha\xi \cos \beta\xi)$	in I $\lambda = \alpha+j\beta$, $\mu = \alpha-j\beta$
III	$m > 0$, $n^4 > 0$, $m = n$; $\psi = \sqrt{(2)}m$	$\cosh \psi\xi - \frac{\psi\xi}{2} \sinh \psi\xi$	$\frac{1}{2\psi}(3 \sinh \psi\xi - \psi\xi \cosh \psi\xi)$	$\frac{1}{2\psi} \xi \sinh \psi\xi$	$\frac{1}{2\psi^3}(\psi\xi \cosh \psi\xi - \sinh \psi\xi)$	in II $\alpha = \psi$, $\beta = 0$
IV	$m > 0$, $n = 0$	1	ξ	$\frac{1}{4m^2}(\cosh 2m\xi - 1)$	$\frac{1}{4m^2}\left(\frac{1}{2m} \sinh 2m\xi - \xi\right)$	in I $\lambda = 2m$, $\mu = 0$

TABLE 1—*continued*

V	$m > 0$, $n^4 < 0$	$\bar{\lambda} = [2m^2 + 2(m^4 + \bar{n}^4)^{\frac{1}{2}}]^{\frac{1}{2}}$ $\bar{\mu} = [-2m^2 + 2(m^4 + \bar{n}^4)^{\frac{1}{2}}]^{\frac{1}{2}}$ $\bar{n} = \frac{\sqrt{2}}{2}(1+j)n$	$\frac{1}{\bar{\mu}^2 + \bar{\lambda}^2}(\bar{\mu}^2 \cosh \bar{\lambda}\xi + \bar{\lambda}^2 \cos \bar{\mu}\xi)$	$\frac{1}{\bar{\mu}^2 + \bar{\lambda}^2}\left(\frac{\bar{\mu}^2}{\bar{\lambda}} \sinh \bar{\lambda}\xi + \frac{\bar{\lambda}^2}{\bar{\mu}} \sin \bar{\mu}\xi\right)$	$\frac{1}{\bar{\mu}^2 + \bar{\lambda}^2}(\cosh \bar{\lambda}\xi - \cos \bar{\mu}\xi)$	$\frac{1}{\bar{\mu}^2 + \bar{\lambda}^2}\left(\frac{1}{\bar{\lambda}} \sinh \bar{\lambda}\xi - \frac{1}{\bar{\mu}} \sin \bar{\mu}\xi\right)$	in I $\lambda = \bar{\lambda}$ $\mu = j\bar{\mu}$
VI	$m = 0$, $n^4 > 0$		$\cosh n\xi \, . \cos n\xi$	$\frac{1}{2n}(\cosh n\xi \, . \sin n\xi + \sinh n\xi \, . \cos n\xi)$	$\frac{1}{2n^2} \sinh n\xi \, . \sin n\xi$	$\frac{1}{4n^3}(\cosh n\xi \, . \sin n\xi - \sinh n\xi \cos n\xi)$	in II $\alpha = \beta = n$
VII	$m = 0$, $n^4 < 0$	$\varepsilon = \sqrt{(2)}\bar{n}$	$\frac{1}{2}(\cosh \varepsilon\xi + \cos \varepsilon\xi)$	$\frac{1}{2\varepsilon}(\sinh \varepsilon\xi + \sin \varepsilon\xi)$	$\frac{1}{2\varepsilon^2}(\cosh \varepsilon\xi - \cos \varepsilon\xi)$	$\frac{1}{2\varepsilon^3}(\sinh \varepsilon\xi - \sin \varepsilon\xi)$	in V $\bar{\lambda} = \bar{\mu} = \varepsilon$
VIII	$m = 0$, $n = 0$		1	ξ	$\frac{1}{2}\xi^2$	$\frac{1}{6}\xi^3$	

of the deflection curve is zero and the total shear force is one-half the concentrated load amplitude

$$\left[\frac{dv}{d\xi}\right]_{\xi=0} = 0 \tag{17}$$

$$Q(0) = -\frac{EI}{l^3}\left\{\left[\frac{d^3v}{d\xi^3}\right]_{\xi=0} - 4m^2\left[\frac{dv}{d\xi}\right]_{\xi=0}\right\} = -\frac{F_0}{2}. \tag{18}$$

At the free end of the beam, the bending moment vanishes, the deflection of the beam equals the deflection of the free surface of the foundation, and the total shear force equals the concentrated foundation pressure

$$M(1) = -\frac{EI}{l^2}\left[\frac{d^2v}{d\xi^2}\right]_{\xi=1} = 0 \tag{19}$$

$$v(1) = v_f(1) \tag{20}$$

$$Q(1) = -\frac{EI}{l^3}\left\{\left[\frac{d^3v}{d\xi^3}\right]_{\xi=1} - 4m^2\left[\frac{dv}{d\xi}\right]_{\xi=1}\right\} = -R_f(1) = \frac{G_0}{l}\left[\frac{dv_f}{d\xi}\right]_{\xi=1}. \tag{21}$$

At infinity, the deflection of the free surface of the foundation must vanish

$$[v_f]_{\xi\to\infty} = 0. \tag{22}$$

The six conditions (17)–(22) permit the determination of the integration constants from equations (11) and (16). The expressions of the deflection, the slope of the deflection curve,

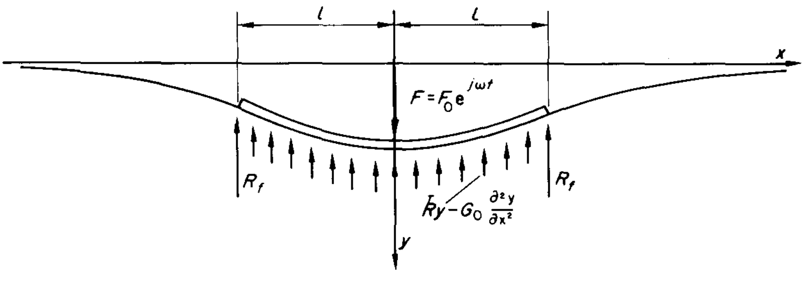

FIG. 2.

the bending moment and the shear force of the beam are the following

$$v(\xi) = -\frac{F_0 l^3}{2EI}[B_1\phi_1(\xi) + B_3\phi_3(\xi) - \phi_4(\xi)] \tag{23}$$

$$\varphi(\xi) = -\frac{F_0 l^2}{2EI}[B_1\phi_1'(\xi) + B_3\phi_3'(\xi) - \phi_4'(\xi)] \tag{24}$$

$$M(\xi) = \frac{F_0 l}{2}[B_1\phi_1''(\xi) + B_3\phi_3''(\xi) - \phi_4''(\xi)] \tag{25}$$

$$T(\xi) = \frac{F_0}{2}[B_1\phi_1'''(\xi) + B_3\phi_3'''(\xi) - \phi_4'''(\xi)], \tag{26}$$

in which primes denote derivatives with respect to ξ, and

$$B_1 = \frac{\phi_3''\phi_4''' - \phi_4''\phi_3''' - 4m^2[\phi_3''\phi_4' - \phi_4''\phi_3' + \gamma(\phi_3''\phi_4 - \phi_4''\phi_3)]}{\phi_3''\phi_1''' - \phi_1''\phi_3''' - 4m^2[\phi_3''\phi_1' - \phi_1''\phi_3' + \gamma(\phi_3''\phi_1 - \phi_1''\phi_3)]} \tag{27}$$

$$B_3 = \frac{\phi_4''\phi_1''' - \phi_1''\phi_4''' - 4m^2[\phi_4''\phi_1' - \phi_1''\phi_4' + \gamma(\phi_4''\phi_1 - \phi_1''\phi_4)]}{\phi_3''\phi_1''' - \phi_1''\phi_3''' - 4m^2[\phi_3''\phi_1' - \phi_1''\phi_3' + \gamma(\phi_3''\phi_1 - \phi_1''\phi_3)]}, \tag{28}$$

where for the sake of brevity the notation $\phi_i = \phi_i(1)$ has been used.

In the following we shall limit ourselves to find the expressions for the beam deflection and the bending moment at the midpoint.

3.1 *Solutions for the deflection at the midpoint*

From equation (23) it follows that the deflection at the middle of the beam is

$$v_0 = v(0) = -\frac{F_0 l^3}{2EI} B_1. \tag{29}$$

For each of the cases mentioned in Table 1, the expressions of v_0 are given below (the corresponding notations are listed in the second column of the table).

Case I. $m > 0,\ n^4 > 0,\ m > n$:

$$v_0 = -\frac{F_0 l^3}{2EI} \frac{\lambda\mu[(\mu^4+\lambda^4)\cosh\lambda\cosh\mu - \mu\lambda(\mu^2+\lambda^2)\sinh\lambda\sinh\mu - 2\mu^2\lambda^2] - \gamma(\mu^4-\lambda^4)(\lambda\cosh\lambda\sinh\mu - \mu\sinh\lambda\cosh\mu)}{\lambda^2\mu^2(\mu^2-\lambda^2)(\lambda^3\cosh\lambda\sinh\mu - \mu^3\sinh\lambda\cosh\mu) - \gamma\lambda\mu(\mu^2+\lambda^2)(\mu^2-\lambda^2)^2\cosh\lambda\cosh\mu} \tag{30}$$

Case II. $m > 0,\ n^4 > 0,\ m < n$:

$$v_0 = \frac{F_0 l^3}{2EI} \frac{(\alpha^2+\beta^2)[\beta^2(\beta^2-3\alpha^2)\sinh^2\alpha - \alpha^2(\alpha^2-3\beta^2)\sin^2\beta - 4\alpha^2\beta^2] - \gamma 2\alpha\beta(\alpha^2-\beta^2)(\beta\sinh 2\alpha - \alpha\sin 2\beta)}{\alpha\beta(\alpha^2+\beta^2)^2[\beta(\beta^2-3\alpha^2)\sinh 2\alpha + \alpha(\alpha^2-3\beta^2)\sin 2\beta] - \gamma 8\alpha^2\beta^2(\alpha^4-\beta^4)(\cosh^2\alpha - \sin^2\beta)}. \tag{31}$$

Case III. $m > 0,\ n^4 > 0,\ m = n$:

$$v_0 = \frac{F_0 l^3}{2EI} \frac{\psi(3\sinh^2\psi + \psi^2 + 4) + 2\gamma(\sinh 2\psi - 2\psi)}{\psi^3[\psi(3\sinh 2\psi - 2\psi) + 8\gamma\cosh^2\psi]}. \tag{32}$$

Case IV. $m > 0,\ n = 0$:

$$v_0 = \frac{F_0 l^3}{2EI} \frac{2m + \gamma(2m - \tanh 2m)}{8\gamma m^3}. \tag{33}$$

Case V. $m > 0,\ n^4 < 0$:

$$v_0 = -\frac{F_0 l^3}{2EI} \frac{\bar\lambda\bar\mu[(\bar\mu^4+\bar\lambda^4)\cosh\bar\lambda\cos\bar\mu + \bar\mu\bar\lambda(\bar\lambda^2-\bar\mu^2)\sinh\bar\lambda\sin\bar\mu + 2\bar\mu^2\bar\lambda^2] - \gamma(\bar\mu^4-\bar\lambda^4)(\bar\lambda\cosh\bar\lambda\sin\bar\mu - \bar\mu\sinh\bar\lambda\cos\bar\mu)}{\bar\mu^2\bar\lambda^2(\bar\mu^2+\bar\lambda^2)(\bar\lambda^3\cosh\bar\lambda\sin\bar\mu + \bar\mu^3\sinh\bar\lambda\cos\bar\mu) - \gamma\bar\lambda\bar\mu(\bar\lambda^2-\bar\mu^2)(\bar\mu^2+\bar\lambda^2)^2\cosh\bar\lambda\cos\bar\mu}. \tag{34}$$

Case VI. $m = 0, n^4 > 0$:

$$v_0 = \frac{F_0 l^3}{2EI} \frac{\cosh 2n + \cos 2n + 2}{4n^3(\sinh 2n + \sin 2n)}. \tag{35}$$

Case VII. $m = 0,\ n^4 < 0$:

$$v_0 = -\frac{F_0 l^3}{2EI} \frac{\cosh \varepsilon \cos \varepsilon + 1}{\varepsilon^3(\cosh \varepsilon \sin \varepsilon + \sinh \varepsilon \cos \varepsilon)}. \tag{36}$$

Substituting $n = \nu(\Omega = 0)$ in equation (35), we obtain the static deflection at the middle of a beam lying on a Winkler-type foundation.

$$v_0^* = \frac{F_0 l^3}{2EI} \frac{\cosh 2\nu + \cos 2\nu + 2}{4\nu^3(\sinh 2\nu + \sin 2\nu)}. \tag{37}$$

3.2 *Solutions for the bending moment at the midpoint*

From equation (25) it is seen that the bending moment at the middle of the beam is

$$M_0 = M(0) = \frac{F_0 l}{2} B_3. \tag{38}$$

For each of the cases considered in Table 1, the corresponding expressions of M_0 are the following:

Case I. $m > 0,\ n^4 > 0,\ m > n$:

$$M_0 = -\frac{F_0 l}{2} \frac{\mu\lambda[(\mu^4 + \lambda^4)\sinh \lambda \sinh \mu + \mu\lambda(\mu^2 + \lambda^2)(1 - \cosh \mu \cosh \lambda)] - \gamma(\mu^4 - \lambda^4)(\lambda \sinh \lambda \cosh \mu - \mu \cosh \lambda \sinh \mu)}{\mu\lambda(\mu^2 - \lambda^2)(\lambda^3 \sinh \mu \cosh \lambda - \mu^3 \sinh \lambda \cosh \mu) - \gamma(\mu^2 + \lambda^2)(\mu^2 - \lambda^2)^2 \cosh \mu \cosh \lambda}. \tag{39}$$

Case II. $m > 0,\ n^4 > 0,\ m < n$:

$$M_0 = \frac{F_0 l}{2} \frac{(\alpha^2 + \beta^2)[\beta^2(\beta^2 - 3\alpha^2)\sinh^2 \alpha + \alpha^2(\alpha^2 - 3\beta^2)\sin^2 \beta] - \gamma \,.\, 2\alpha\beta(\alpha^2 - \beta^2)(\beta \sinh 2\alpha + \alpha \sin 2\beta)}{\alpha\beta(\alpha^2 + \beta^2)[\beta(\beta^2 - 3\alpha^2)\sinh 2\alpha + \alpha(\alpha^2 - 3\beta^2)\sin 2\beta] - \gamma 8\alpha^2\beta^2(\alpha^2 - \beta^2)(\cosh^2 \alpha - \sin^2 \beta)}. \tag{40}$$

Case III. $m > 0,\ n^4 > 0,\ m = n$:

$$M_0 = \frac{F_0 l}{2} \frac{\psi(\psi^2 - 3\sinh^2 \psi) - 2\gamma(\sinh 2\psi + 2\psi)}{\psi^2(2\psi - 3\sinh 2\psi) - \gamma 8\psi \cosh^2 \psi}. \tag{41}$$

Case IV. $m > 0,\ n = 0$:

$$M_0 = \frac{F_0 l}{2} \frac{\tanh 2m}{2m}. \tag{42}$$

Case V. $m > 0,\ n^4 < 0$:

$$M_0 = \frac{F_0 l}{2} \frac{\bar{\lambda}\bar{\mu}[(\bar{\mu}^4 + \bar{\lambda}^4)\sinh \bar{\lambda} \sin \bar{\mu} + \bar{\mu}\bar{\lambda}(\bar{\lambda}^2 - \bar{\mu}^2)(1 - \cosh \bar{\lambda} \cos \bar{\mu})] + \gamma(\bar{\mu}^4 - \bar{\lambda}^4)(\bar{\lambda} \sinh \bar{\lambda} \cos \bar{\mu} + \bar{\mu} \cosh \bar{\lambda} \sin \bar{\mu})}{\bar{\lambda}\bar{\mu}(\bar{\mu}^2 + \bar{\lambda}^2)(\bar{\lambda}^3 \sin \bar{\mu} \cosh \bar{\lambda} + \bar{\mu}^3 \sinh \bar{\lambda} \cos \bar{\mu}) + \gamma(\bar{\mu}^2 - \bar{\lambda}^2)(\bar{\mu}^2 + \bar{\lambda}^2)^2 \cosh \bar{\lambda} \cos \bar{\mu}}. \tag{43}$$

Case VI. $m = 0$, $n^4 > 0$:

$$M_0 = \frac{F_0 l}{2} \frac{\cosh 2n - \cos 2n}{2n(\sinh 2n + \sin 2n)} \tag{44}$$

Case VII. $m = 0$, $n^4 < 0$:

$$M_0 = \frac{F_0 l}{2} \frac{\sinh \varepsilon \,.\, \sin \varepsilon}{\varepsilon(\cosh \varepsilon \sin \varepsilon + \sinh \varepsilon \,.\, \cos \varepsilon)}. \tag{45}$$

Case VIII. $m = 0$, $n = 0$:

$$M_0 = \frac{F_0 l}{4}. \tag{46}$$

The static bending moment M_0^* at the middle of a beam resting on a Winkler-type foundation can be obtained from equation (44), setting $n = \nu(\Omega = 0)$

$$M_0^* = \frac{F_0 l}{2} \frac{\cosh 2\nu - \cos 2\nu}{2\nu(\sinh 2\nu + \sin 2\nu)}. \tag{47}$$

3.3 *Frequency-response curves*

In plotting the response curves, it is desirable to express the beam deflections and the bending moments by dimensionless quantities, using as reference values the expressions (37) and (47). Each of the relations (30)–(46) deduced above, permit the computation of the response only inside a limited range of forcing frequencies, depending upon the relative magnitude of the parameters m and ν, as shown in Table 2.

In Fig. 3 and Fig. 4, the displacement and bending moment at the middle of the beam are plotted as functions of Ω, for $\nu = \sqrt{(2)}/2$, $m = 0$ (broken line) and $m = 0{\cdot}5$ (solid line).

Concerning the displacement amplitudes (Fig. 3), for both foundation models, the steady-state response of the spring-supported beam shows unbounded values in the neighbourhood of the "rigid-body bouncing" frequency, and at frequencies corresponding to the resonances of an unsupported vibrating beam. The resonant frequencies of the beam on a Pasternak foundation are higher than those of the beam lying on a Winkler foundation, the shear layer having a stiffening effect (equivalent to that of an axial tensile force in the beam). At high forcing frequencies the beam responses are similar for both supporting foundations.

Regarding the bending moments in the beam (Fig. 4), the responses differ substantially, especially in the vicinity of the "rigid-body bouncing frequency," where the Pasternak foundation gives rise to an interesting resonance. This reveals an insufficiency of Winkler's model. At the first resonant frequency, owing to the lack of an interaction between the loaded and the free surface of the foundation, the beam moves with large vertical amplitude but without bending much (like a rigid body), so that the dynamic amplification of the bending moments is insignificant.

The unbounded growth of stresses in a beam on a Pasternak foundation at the "bouncing frequency" is due to the introduction of the shear layer. The effect of the foundation part situated outside the beam appears as two concentrated reactions at its ends, having the same action as two elastic lumped supports of constant $(\gamma G_0)/l$; practically, it means a change in the boundary conditions which make boundless displacements to be accompanied by boundless bending moments.

TABLE 2.

Model	Frequency range				Case in Table 1		Analytical expression	
	$m = 0$	$m > \nu$	$m = \nu$	$m < \nu$			Bending moment	Deflection
Pasternak				$0 \leq \Omega < \left[1-\left(\frac{m}{\nu}\right)^4\right]^{\frac{1}{2}}$	II	$n^4 > 0$ $m < n$	(40)	(31)
			$\Omega = 0$	$\Omega = \left[1-\left(\frac{m}{\nu}\right)^4\right]^{\frac{1}{2}}$	III	$n^4 > 0$ $m = n$	(41)	(32)
		$0 \leq \Omega < 1$	$0 < \Omega < 1$	$\left[1-\left(\frac{m}{\nu}\right)^4\right]^{\frac{1}{2}} < \Omega < 1$	I	$n^4 > 0$ $m > n$	(39)	(30)
		$\Omega = 1$	$\Omega = 1$	$\Omega = 1$	IV	$n = 0$	(42)	(33)
		$1 < \Omega$	$1 < \Omega$	$1 < \Omega$	V	$n^4 < 0$	(43)	(34)
Winkler	$\Omega < 1$				VI	$n^4 > 0$	(44)	(35)
	$1 < \Omega$				VII	$n^4 < 0$	(45)	(36)

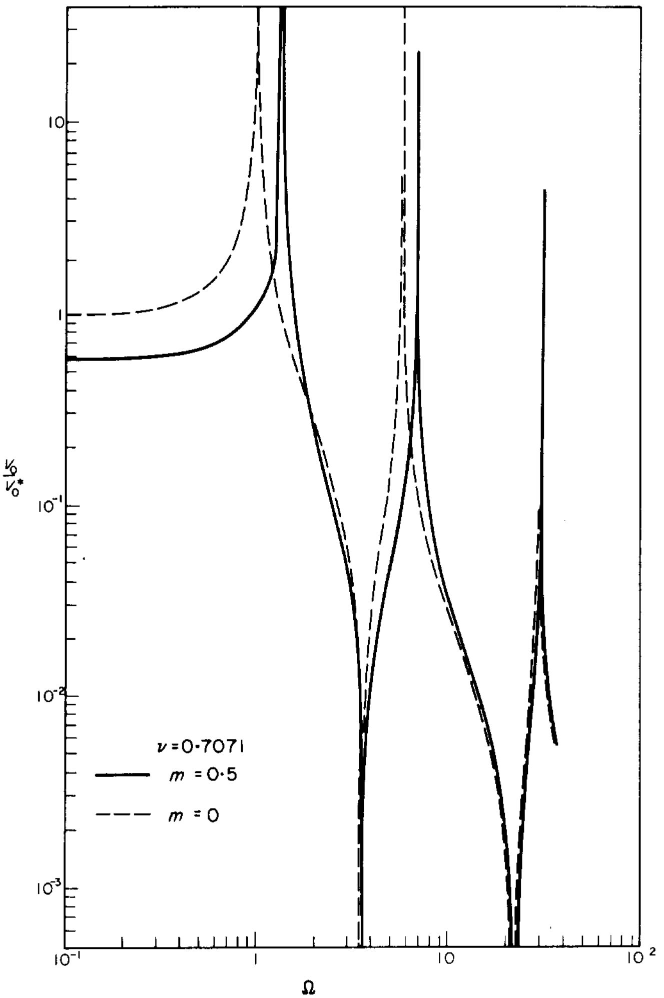

FIG. 3.

The response curves are plotted only for a restricted range of forcing frequencies. For forcing frequencies higher than the third critical frequency, the simplified Bernoulli–Euler beam model limits the applicability of the present method; a Timoshenko beam must be introduced, and the foundation inertia cannot be neglected.

Attention must be paid to the following remark. While for low forcing frequencies, the response of a beam supported by a Winkler foundation can be calculated accurately by considering the beam to be completely rigid, in the case of a beam resting on a Pasternak foundation, the influence of the shear layer and of the corresponding concentrated pressures at the ends are dependent on the beam curvature, wherefrom it follows that the assumption of rigidity could lead to errors. Indeed, from Fig. 5 one may see that the bending moments in a rigid beam (broken line) are greater than in an elastic one (solid line).

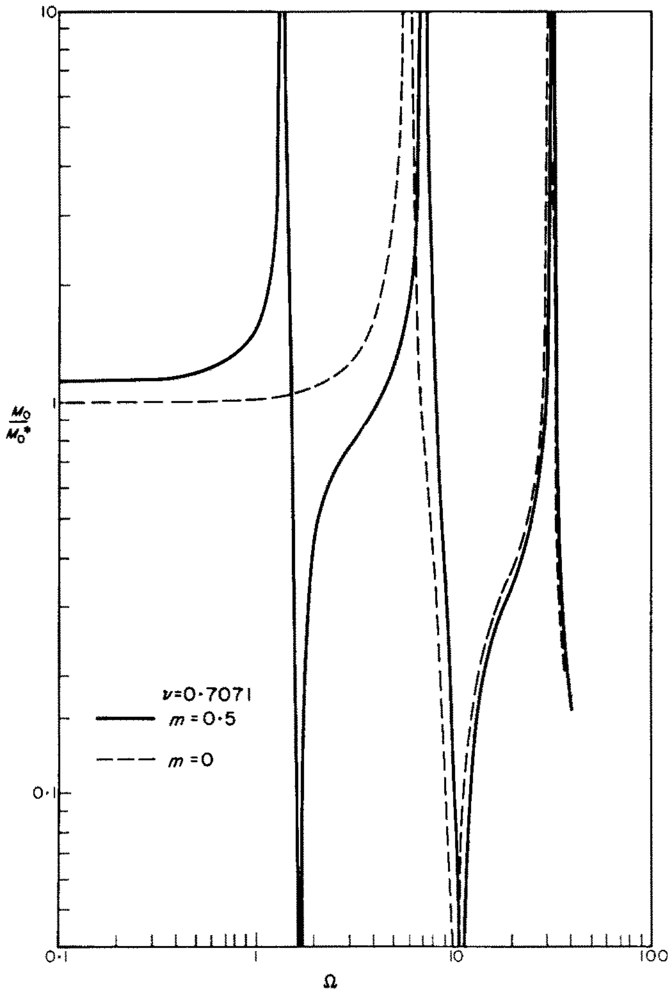

FIG. 4.

4. FREE-FREE BEAM LOADED AT THE MIDDLE BY A COUPLE

A couple $\mathscr{M} = \mathscr{M}_0\, e^{j\omega t}$, acting at the middle of the beam, induces antisymmetric vibrations. Considering only the right side of a finite free-free beam and taking the origin of coordinates at the midpoint, the boundary conditions (19)–(22) are the same as for the symmetric case. On the contrary, at the origin, the deflection is zero and the bending moment is one-half the amplitude of the couple

$$[v]_{\xi=0} = 0 \tag{17'}$$

$$M(0) = -\frac{EI}{l^2}\left[\frac{d^2v}{d\xi^2}\right]_{\xi=0} = \frac{\mathscr{M}_0}{2}. \tag{18'}$$

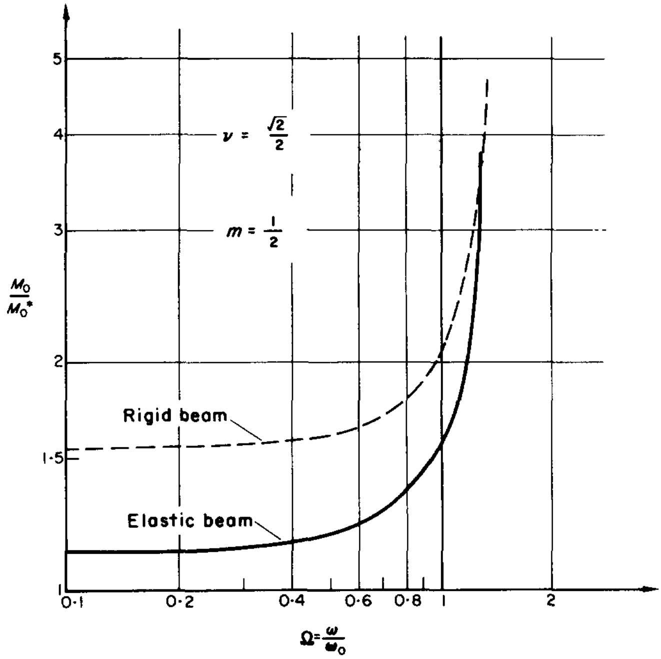

FIG. 5.

The expressions for deflection, slope of the deflection curve, bending moment and shear force are the following

$$v(\xi) = \frac{\mathcal{M}_0 l^2}{2EI}[B_2\phi_2(\xi)+B_4\phi_4(\xi)-\phi_3(\xi)] \tag{23'}$$

$$\varphi(\xi) = \frac{\mathcal{M}_0 l}{2EI}[B_2\phi_2'(\xi)+B_4\phi_4'(\xi)-\phi_3'(\xi)] \tag{24'}$$

$$M(\xi) = -\frac{\mathcal{M}_0}{2}[B_2\phi_2''(\xi)+B_4\phi_4''(\xi)-\phi_3''(\xi)] \tag{25'}$$

$$T(\xi) = -\frac{\mathcal{M}_0}{2l}[B_2\phi_2'''(\xi)+B_4\phi_4'''(\xi)-\phi_3'''(\xi)] \tag{26'}$$

in which

$$B_2 = \frac{(\phi_4''\phi_3'''-\phi_3''\phi_4''')-4m^2(\phi_4''\phi_3'-\phi_3''\phi_4')-\gamma 4m^2(\phi_4''\phi_3-\phi_3''\phi_4)}{(\phi_4''\phi_2'''-\phi_2''\phi_4''')-4m^2(\phi_4''\phi_2'-\phi_2''\phi_4')-\gamma 4m^2(\phi_4''\phi_2-\phi_2''\phi_4)} \tag{27'}$$

$$B_4 = \frac{(\phi_3''\phi_2'''-\phi_2''\phi_3''')-4m^2(\phi_3''\phi_2'-\phi_2''\phi_3')-\gamma 4m^2(\phi_3''\phi_2-\phi_2''\phi_3)}{(\phi_4''\phi_2'''-\phi_2''\phi_4''')-4m^2(\phi_4''\phi_2'-\phi_2''\phi_4')-\gamma 4m^2(\phi_4''\phi_2-\phi_2''\phi_4)} \tag{28'}$$

where $\phi_i = \phi_i(1)$.

5. PASTERNAK FOUNDATION MODEL INCLUDING INERTIA

At high exciting frequencies, a third term must be added to the expression of the foundation reaction, namely, a component taking into account its inertia. This can be accomplished by assuming a mass $\varrho_0 A_0$ for each vertical element of the shear layer attached to the ends of foundation springs.

Instead of equation (2) we introduce

$$q(x, t) = ky - G_0 \frac{\partial^2 y}{\partial x^2} + \varrho_0 A_0 \frac{\partial^2 y}{\partial t^2}. \tag{48}$$

For practical values of the forcing frequency ($\Omega < \zeta^{-\frac{1}{2}}$), the beam response can be calculated using the above deduced expressions, but introducing

$$\bar{\gamma} = \gamma(1 - \zeta\Omega^2)^{\frac{1}{2}} \tag{49}$$

instead of γ, and

$$\hat{n} = \nu[1 - (1 + \zeta)\Omega^2]^{\frac{1}{4}} \tag{50}$$

instead of n. In (49) and (50) the following notation has been used

$$\zeta = \frac{\varrho_0 A_0}{\varrho A}. \tag{51}$$

6. PASTERNAK FOUNDATION INCLUDING DAMPING

Based on the Alfrey–Lee principle of correspondence [17] and according to Berry [18], the solution of the viscoelastic problem for a steady-state harmonic excitation, can be obtained directly from that of the corresponding elastic problem, by replacing the real moduli and constants of elasticity by complex moduli, generally dependent on the frequency. Thus, the steady-state response of a viscoelastic Pasternak foundation can be described by "complex foundation moduli"

$$k^* = k(1 + j\delta_k) \tag{52}$$

and

$$G_0^* = G_0(1 + j\delta_G), \tag{53}$$

where k and G_0 are dynamic foundation moduli, δ_k and δ_G are damping factors of the foundation. The response of a beam resting on a foundation of any viscoelastic material can be computed using for each value of the forcing frequency the corresponding values of k, G_0, δ_k, δ_G—experimentally obtained.

Introducing the complex Young's modulus of the beam material

$$E^* = E(1 + j\eta), \tag{54}$$

the governing equation (5) becomes

$$\frac{d^4 v}{d\xi^4} - 4m^{*2} \frac{d^2 v}{d\xi^2} + 4n^{*4} v = 0, \tag{55}$$

where $v(\xi)$ is a complex amplitude of the beam deflection, and

$$m^* = m\left(\frac{1+j\delta_G}{1+j\eta}\right)^{\frac{1}{2}} \tag{56}$$

$$n^* = n\left(\frac{1+j\dfrac{\delta_k}{1-\Omega^2}}{1+j\eta}\right)^{\frac{1}{4}}, \tag{57}$$

in which m and n are defined by equations (6) and (7).

Equation (14) becomes

$$\frac{\mathrm{d}^2 v_f}{\mathrm{d}\xi^2} - \gamma^{*2} v_f = 0, \tag{58}$$

in which

$$\gamma^* = \gamma\left(\frac{1+j\delta_k}{1+j\delta_G}\right)^{\frac{1}{2}}, \tag{59}$$

where γ results from equation (15).

The solution of the problem is also given by equations (30)–(46); this time v_0 and M_0 are complex quantities, the absolute values of which must be calculated.

For example, in Case I (Table 1), in equations (30) and (39), μ and λ are to be replaced by

$$\begin{aligned} \lambda^* &= (m^{*2}+n^{*2})^{\frac{1}{2}} + (m^{*2}-n^{*2})^{\frac{1}{2}} \\ \mu^* &= (m^{*2}+n^{*2})^{\frac{1}{2}} - (m^{*2}-n^{*2})^{\frac{1}{2}} \end{aligned} \tag{60}$$

and γ by γ^* (59). The absolute values of v_0 and M_0 can then be obtained by routine means.

Generally, the damping factor η has a significant influence only over the first resonant frequency of the unsupported beam, δ_k—only in the vicinity of the "rigid-body bouncing" and δ_G—in a wide range of intermediate and high frequencies.

In Fig. 6 are given response curves plotted only for the range of low forcing frequencies and for different values of the damping factors. The deflections are normalized by division by the static deflection at the middle of the beam supported by a Pasternak foundation.

7. CONCLUSIONS

The method of initial parameters seems to be a convenient means of analysis of steady-state responses of beams on deformable subgrades.

At low forcing frequencies, the beam response is affected by introduction of the "shear layer" in the Pasternak model, which brings forth an important resonance for the bending moments in the vicinity of the bouncing frequency of the rigid beam. When the forcing frequency grows, the effect of the shear layer diminishes and the response becomes similar to that of the beam on a Winkler foundation.

The influence of linear foundation damping on the response is exhibited by introducing complex foundation moduli, the solution being thus independent of the choice of a particular rheological model.

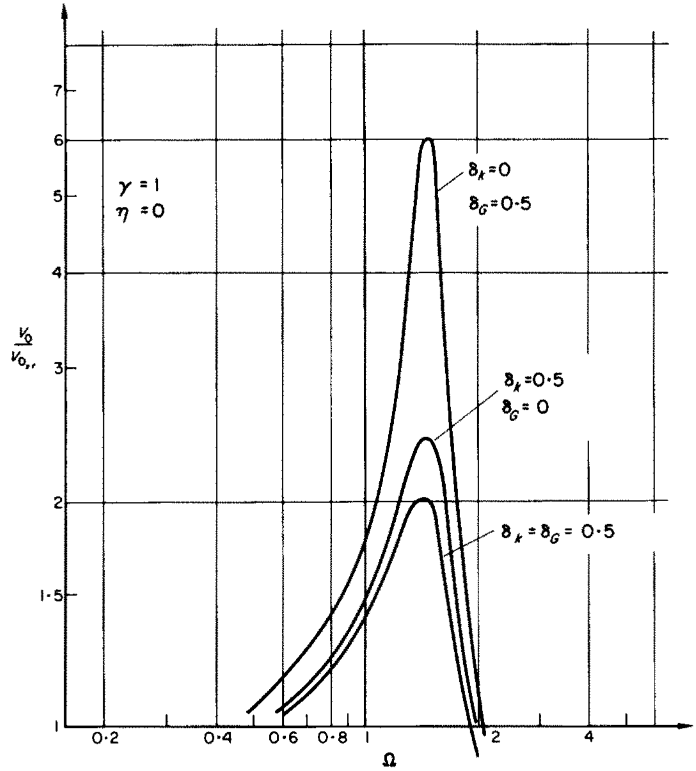

FIG. 6.

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Абстракт—Постоянная балка Бернулли-Зйлера формы конечной длины, опертая на основании типа Пастернака подвергается действию гармонической силы $F = F_0 e^{j\omega t}$, приложенной к серединной точке. Показано влияние "слоя сдвига" по сравнению с поведением балки, лежащей на основании типа Винклера. При первой частоте резонанса, появляются неограниченные значения момента изгиба, которые не встречаются как зто предусматривает теория Винклера. Наблюдается влияние демпфирования и инерции основания. Даются так аналитические зависимости, как и кривые частот для прогибов балки и моментов изгиба.